The Computation of the Kronecker Canonical Form of an Arbitrary Symmetric Pencil

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Submitted by Ludwig Elsner

ABSTRACT

We present an algorithm for the computation of the Kronecker structure of a symmetric pencil $A - \lambda M$. This method, which preserves the property of symmetry, is somewhat different from Van Dooren's algorithm for the determination of the Kronecker structure of an arbitrary pencil. We show how to use this method to determine the structure of the infinite elementary divisors of $A - \lambda M$ and its Kronecker blocks, which may occur for the case of a singular pencil. The cost of computations of our algorithm is cheaper than Van Dooren's algorithm, and the symmetry of the reduced pencils can be preserved. The algorithm is fairly stable, though we use some nonunitary transformation matrices, but the norms of these matrices are bounded under a tolerance. The present procedure can also be used to separate from a symmetric pencil a smaller symmetric regular pencil, which contains only the finite eigenvalues of the original one; so this method can be used as a "preprocessing" for a QZ or HR algorithm in order to get rid of singularities, which can be troublesome for those algorithms.

1. INTRODUCTION

In this paper we investigate the problem of determining the eigenstructure (eigenvalues and associated Kronecker structure [6, p. 29]) of a square symmetric $n \times n$ singular pencil $A - \lambda M$ over $\mathbb{R}$. By definition [6] a square pencil $C - \lambda B$ [or a pair (C, B)] is called regular if $\det(C - \lambda B) \not\equiv 0$. In the case $\det(C - \lambda B) \equiv 0$ the pencil is called singular. Two square pencils $C - \lambda B$ and $C_1 - \lambda B_1$ are said to be strictly equivalent if there exist two

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constant invertible matrices P and Q such that

$$P(C-\lambda B)Q = C_1 - \lambda B_1. \tag{1.1}$$

We sometimes denote this equivalence relation by $\sim$.

P and Q are called respectively the left and right transformation matrices of this equivalent transformation. Kronecker's theory of singular pencils [6, p. 41] shows that any symmetric pencil $A-\lambda M$ is strictly equivalent to the symmetric pencil

$$\operatorname{diag}\left\{0, \begin{bmatrix} 0 & L_{\epsilon_1}^T \\ L_{\epsilon_1} & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & L_{\epsilon_p}^T \\ L_{\epsilon_p} & 0 \end{bmatrix}, N_1, \dots, N_s, Z_1, \dots, Z_r\right\}, \tag{1.2}$$

where

(1) L_k is the $k \times (k+1)$ bidiagonal pencil (left Kronecker block)

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \lambda & 1 \end{bmatrix},$$

(2) L_k^T is the $(k+1)\times k$ bidiagonal pencil (right Kronecker block)

$$\begin{bmatrix} \lambda & & 0 \\ 1 & \ddots & \\ & \ddots & \lambda \\ 0 & & 1 \end{bmatrix},$$

(3) N_k has the form

$$\begin{bmatrix} 0 & & & \lambda & 1 \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ \lambda & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & 0 \end{bmatrix}$$

associated to the nilpotent Jordan block (infinite elementary divisor),

(4) Z_k has the form

$$\begin{bmatrix} 0 & & & 1 & \lambda+\alpha \\ & & \cdot & \cdot & \\ & \cdot & \cdot & & \\ 1 & \cdot & & & \\ \lambda+\alpha & & & & 0 \end{bmatrix}$$

associated to the Jordan canonical block (finite elementary divisor of α).

Several numerical algorithms [5, 7, 8, 10–12] have been elaborated for the problem of determining the Kronecker structure of an arbitrary pencil. But when the pencil is symmetric, these algorithms destroy the symmetry of the currently reduced pencil after the first rank compression (see e.g. [8, 10]). Fix and Heiberger [3] have described an algorithm to cancel the singularity of $A-\lambda M$ for a positive semidefinite matrix M and reduce the problem to a small symmetric pencil $\tilde{A}-\lambda I$, which possesses only finite eigenvalues. In this paper we generalize Fix and Heiberger's reducing procedure on a symmetric pencil $A-\lambda M$ to separate all singularities [including nilpotent Jordan blocks and left and right Kronecker blocks in (1.2)] and the "finite" symmetric regular pencil $A_f-\lambda M_f$, which has only finite eigenvalues of $A-\lambda M$. The symmetry of the currently reduced pencil in this procedure is preserved. One may now make use of the QZ [9] or HR [1, 2] algorithms on this small regular pencil $A_f-\lambda M_f$ to compute the finite eigenvalues. If a finite eigenvalue α is known, the above algorithm can be performed on a new symmetric pencil $M-\lambda(A-\alpha M)$ to determine the complete Jordan structure associated to α. In Section 2 we give a computational algorithm for symmetric regular pencils; the cost of computations and the numerical stability are also discussed. The generalized eigenvalue problem for symmetric singular pencils is given in Section 3. It is similar to the previous algorithm, but more complicated, since Kronecker blocks may also occur [6].

Throughout this paper we denote the unit matrix of size n by I_n, the $n\times n$ and $m\times n$ zero matrices by 0_n and $0_{m,n}$ respectively, the conjugate transpose and transpose of A by $A^H\equiv\bar{A}^T$ and A^T respectively, and the submatrix of A which consists of the i_1th to i_2th rows and j_1th to j_2th columns by $A(i_1,i_2;j_1,j_2)$. The direct sum of two matrices A and B is denoted by $A\oplus B$.

Preliminaries

We frequently use the following procedures to determine the rank of a given matrix. Let η be a user-given criterion for negligibility; the calculation should be made with two different values of η (say $\eta = n\epsilon$ or $\eta = \sqrt{\epsilon}$, where ϵ = machine precision; see [16, p. 313]).

(i) Let M be a real symmetric $n \times n$ matrix. There exists an orthogonal matrix Q [14, p. 227] such that

$$M = Q\Phi Q^T,$$

where

$$\Phi = \operatorname{diag}(\varphi_1, \dots, \varphi_r, \dots, \varphi_n) \equiv \Phi_1 \oplus \Phi_2 \qquad (\Phi_1 \in \mathbb{R}^{r \times r})$$

with

$$|\varphi_1| \geqslant |\varphi_2| \geqslant \cdots \geqslant |\varphi_r| > \eta \geqslant |\varphi_{r+1}| \geqslant \cdots \geqslant |\varphi_n|.$$

Replacing Φ_2 by 0_{n-r}, we have

$$M_\eta = Q(\Phi_1 \oplus 0_{n-r})Q^T \qquad \text{with} \quad \operatorname{rank}(M_\eta) = r$$

and

$$\|M - M_\eta\|_2 \leqslant \eta.$$

We define the spectral decomposition of M by

$$Q^T M Q := (\Phi_1 \oplus 0_{n-r}). \tag{1.3}$$

(ii) Let B be a real (complex) $m \times n$ matrix, $m \geqslant n$. There exist $m \times m$ and $n \times n$ orthogonal (unitary) matrices U and V [4] such that

$$B = U \begin{bmatrix} \Sigma & 0 \\ 0 & \Omega \\ 0 & 0 \end{bmatrix} V^H,$$

where

$$\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r) \quad \text{and} \quad \Omega = (\omega_{r+1}, \dots, \omega_n)$$

with

$$\sigma_1 \geqslant \cdots \geqslant \sigma_r > \eta \geqslant \omega_{r+1} \geqslant \cdots \geqslant \omega_n \geqslant 0.$$

Set $\Omega = 0_{n-r}$; we then have the singular value decomposition (SVD) of a rank r matrix B_η (say):

$$B_\eta = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} V^H \qquad \text{with} \quad \operatorname{rank}(B_\eta) = r,$$

and

$$\|B - B_\eta\| \leqslant \eta.$$

Define the SVD of B by

$$U^T B V := \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.4}$$

(iii) Let C be a complex symmetric $n \times n$ matrix ($C^T = C$). Then from [4, 13] we have a decomposition of C

$$V^T C V = \begin{bmatrix} \Sigma & 0 \\ 0 & \Omega \end{bmatrix} \}\, r \qquad (V \text{ unitary})$$

with

$$\sigma_1 \geqslant \cdots \geqslant \sigma_r > \eta \geqslant \omega_{r+1} \geqslant \cdots \geqslant \omega_n \geqslant 0.$$

Replacing Ω by 0_{n-r}, we have the SVD of a rank r matrix C_η (say):

$$V^T C_\eta V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{with} \quad \operatorname{rank}(C_\eta) = r,$$

and

$$\|C - C_\eta\| \leqslant \eta.$$

Define the SVD of C by

$$V^TCV := \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.5}$$

2. SYMMETRIC REGULAR PENCILS

In this section we will compute the eigenstructure of the real symmetric $n\times n$ regular pencil $A-\lambda M$ using similarity transformations simultaneous on both matrices A and M in order to preserve symmetry of some submatrices of the original pencil. We first determine the eigenstructure associated to an infinite eigenvalue. If a finite eigenvalue α is known, we consider a new pencil $M-\lambda(A-\alpha M)$; using this shift, the eigenstructure corresponding to α can thus be treated as above. We now perform the following steps (set $A_1 := A$, $M_1 := M$, and $n_1 := n$):

Step 1. (a) Find the spectral decomposition of M_1 [as (1.3)]:

$$Q_1^TM_1Q_1 = \mathring{M}_1 \equiv \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{(with nullity } s_1\text{)}, \tag{2.0}$$

where Λ_1 is an $(n_1-s_1)\times(n_1-s_1)$ nonzero diagonal matrix. Multiply the matrix A_1 by Q_1^T and Q_1 on the left and right respectively, and partition the transformed matrix $\mathring{A}_1$ in the following form:

$$Q_1^TA_1Q_1 = \mathring{A}_1 \equiv \begin{array}{c} \begin{bmatrix} W_1 & C_1 \\ C_1^T & H_1 \end{bmatrix} \end{array} \begin{array}{l} \}\, n_1-s_1 \\ \}\, s_1 \end{array}. \tag{2.1}$$

(b) Find the spectral decomposition of H_1 [as in (1.3)]:

$$\tilde{P}_1^TH_1\tilde{P}_1 = \begin{bmatrix} \Phi_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{l} \}\, a_1 \\ \}\, s_2 \end{array}, \tag{2.2}$$

where Φ_1 is an $a_1\times a_1$ nonzero diagonal matrix and $a_1+s_2=s_1$. Multiply $\mathring{A}_1$ and $\mathring{M}_1$ by $P_1^T \equiv \operatorname{diag}(I_{n_1-s_1}, \tilde{P}_1^T)$ and P_1 on the left and right respectively, and partition as follows:

$$P_1^T\mathring{A}_1P_1=\tilde{A}_1\equiv\begin{bmatrix}\tilde{W}_1 & \tilde{C}_1 & \tilde{G}_1\\ \tilde{C}_1^T & \Phi_1 & 0\\ \tilde{G}_1^T & 0 & 0\end{bmatrix}\begin{matrix}\}\,n_1-s_1\\ \}\,a_1\\ \}\,s_2\end{matrix}$$

$$P_1^T\mathring{M}_1P_1=\tilde{M}_1\equiv\begin{bmatrix}\Lambda_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{matrix}\}\,n_1-s_1\\ \}\,a_1\\ \}\,s_2\end{matrix}.$$

(c) Compute the singular value decomposition of $\tilde{G}_1$ [as in (1.4)],

$$\hat{U}_1^T\tilde{G}_1\hat{V}_1=\begin{bmatrix}\Theta_1\\ 0\end{bmatrix}\}\,s_2. \tag{2.3}$$

Here $\tilde{G}_1$ has full column rank (i.e., Θ_1 is nonsingular), since if it did not, that would imply that $\det(A-\lambda M)\equiv 0$. Multiply $\tilde{A}_1$ and $\tilde{M}_1$ by $U_1^T\equiv\operatorname{diag}(\hat{U}_1^T, I_{s_1})$ and $V_1^T\equiv\operatorname{diag}(I_{n_1-s_2},\hat{V}_1^T)$ on the left and by U_1, V_1 on the right. The resulting matrices are then partitioned as

$$V_1^TU_1^T\tilde{A}_1U_1V_1=\hat{A}_1\equiv\begin{bmatrix}\hat{H}_1 & \hat{C}_1 & \Pi_1 & \Theta_1\\ \hat{C}_1^T & A_3 & \hat{D}_1 & 0\\ \Pi_1^T & \hat{D}_1^T & \Phi_1 & 0\\ \Theta_1 & 0 & 0 & 0\end{bmatrix}\begin{matrix}\}\,s_2\\ \}\,n_3\\ \}\,a_1\\ \}\,s_2\end{matrix}$$

and

$$V_1^TU_1^T\tilde{M}_1U_1V_1=\hat{M}_1\equiv\begin{bmatrix}\hat{K}_1 & \hat{G}_1 & 0 & 0\\ \hat{G}_1^T & M_3 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{matrix}\}\,s_2\\ \}\,n_3\\ \}\,a_1\\ \}\,s_2\end{matrix},$$

where

$$\begin{bmatrix} \hat{H}_1 & \hat{C}_1 \\ \hat{C}_1^T & A_3 \end{bmatrix} \equiv \hat{U}_1^T \tilde{W}_1 \hat{U}_1, \qquad \begin{bmatrix} \hat{K}_1 & \hat{G}_1 \\ \hat{G}_1^T & M_3 \end{bmatrix} \equiv \hat{U}_1^T \Lambda_1 \hat{U}_1,$$

$$\tilde{C}_1^T \hat{U}_1 = [\Pi_1^T | \hat{D}_1^T],$$

$n_2 := n_1 - s_1$, and $n_3 = n_2 - s_2$.

(d) Eliminate $\hat{L}_1$ with Φ_1 as row pivot; i.e., there exists a row transformation F_1^T that uses Φ_1 to zero out $\hat{D}_1$. We can partition the matrices as follows:

$$F_1^T \hat{A}_1 F_1 = \left[\begin{array}{cc|cc} \hat{H}_1 & \hat{B}_1 & \Pi_1 & \Theta_1 \\ \hat{B}_1^T & A_3 & 0 & 0 \\ \hline \Pi_1^T & 0 & \Phi_1 & 0 \\ \Theta_1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, s_2 \\ \}\, n_3 \\ \}\, a_1 \\ \}\, s_2 \end{array} \tag{2.4a}$$

with

$$\hat{B}_1^T := \hat{C}_1^T - \hat{D}_1 \Phi_1^{-1} \Pi_1^T,$$

$$A_3 := A_3 - \hat{D}_1 \Phi_1^{-1} \hat{D}_1^T;$$

$$F_1^T \hat{M}_1 F_1 = \left[\begin{array}{cc|cc} \hat{K}_1 & \hat{G}_1 & 0 & 0 \\ \hat{G}_1^T & M_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, s_2 \\ \}\, n_3 \\ \}\, a_1 \\ \}\, s_2 \end{array}, \tag{2.4b}$$

where

(i) Φ_1 and Θ_1 have full rank a_1 and s_2, respectively, and $s_1 = a_1 + s_2$.

(ii) $(\hat{K}_1, \hat{G}_1)$ and $(\hat{G}_1^T, M_3)$ have full row rank s_2 and n_3 respectively.

We denote by X_1 the product of the transformation matrices $Q_1 P_1 U_1 V_1 F_1$.

Step 2. Repeat the above procedure (a)–(d) on the symmetric pencil $A_3-\lambda M_3$ with $n_3\times n_3$ transformation matrices $\hat X_3=Q_3P_3U_3V_3F_3$, thus obtaining a similar reduction to (2.4) of this smaller pencil. These transformations may be embedded in $X_3=\operatorname{diag}(I_{s_2},\hat X_3, I_{s_1})$, so that we have

$$X_3^TX_1^TA_1X_1X_3=\left[\begin{array}{c|cccc|c|c}
\hat H_1 & \multicolumn{4}{c|}{\hat B_1} & \Pi_1 & \Theta_1 \\ \hline
 & \hat H_3 & \hat B_3 & \Pi_3 & \Theta_3 & & \\
\hat B_1^T & \hat B_3^T & A_5 & 0 & 0 & 0 & 0 \\
 & \Pi_3^T & 0 & \Phi_3 & 0 & & \\
 & \Theta_3 & 0 & 0 & 0 & & \\ \hline
\Pi_1^T & \multicolumn{4}{c|}{0} & \Phi_1 & 0 \\ \hline
\Theta_1 & \multicolumn{4}{c|}{0} & 0 &
\end{array}\right]\begin{array}{l}\}\,s_2\\ \}\,s_4\\ \}\,n_5\\ \}\,a_3\\ \}\,s_4\\ \}\,a_1\\ \}\,s_2\end{array}\tag{2.5a}$$

with $\hat B_1:=\hat B_1\hat X_3$, and

$$X_3^TX_1^TM_1X_1X_3=\left[\begin{array}{c|cc|cc|c|c}
\hat K_1 & \multicolumn{2}{c|}{\hat G_{1,1}} & \multicolumn{2}{c|}{\hat G_{1,2}} & 0 & 0 \\ \hline
\hat G_{1,1}^T & \hat K_3 & \hat G_3 & 0 & 0 & 0 & 0 \\
 & \hat G_3^T & M_5 & 0 & 0 & & \\ \hline
\hat G_{1,2}^T & 0 & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & 0 & & \\ \hline
0 & \multicolumn{4}{c|}{0} & 0 & 0 \\ \hline
0 & \multicolumn{4}{c|}{0} & 0 & 0
\end{array}\right]\begin{array}{l}\}\,s_2\\ \}\,s_4\\ \}\,n_5\\ \}\,a_3\\ \}\,s_4\\ \}\,a_1\\ \}\,s_2\end{array}\tag{2.5b}$$

with $\left[\hat G_{1,1}|\hat G_{1,2}\right]:=\hat G_1\hat X_3$, where

(i) Φ_3 and Θ_3 have full rank a_3 and s_4 respectively, and $a_3+s_4=s_3$;

(ii) $\left[\hat K_1,\hat G_{1,1},\hat G_{1,2}\right]$, $\left[\hat K_3,\hat G_3\right]$, and $\left[\hat G_3^T, M_5\right]$ have full row rank s_2, s_4, and n_5 respectively; and

(iii) $\hat G_{1,2}$ have full column rank s_3.

Step j (induction step). Repeat this procedure until the matrix M_j (for some j) has full rank. We write this procedure in the following algorithm:

ALGORITHM 2.1.

(1) $j := 1$; $A_1 := A$; $M_1 := M$; $n_1 := n$; $X := I_n$; $u := 0$; $v := 0$ ($a_0 = s_{-1} = s_0 = 0$).

(2) repeat the procedure (a)–(d) as in step 1 on the $n_j \times n_j$ matrix pencil $A_j - \lambda M_j$; compute the transformation matrix $\hat{X}_j := Q_j P_j U_j V_j F_j$; let $s_j :=$ nullity of M_j; $a_j :=$ rank of Φ_j; $a_{j-1} := s_{j-1} - s_j$; $s_{j+1} :=$ rank of Θ_j;

(3) if $s_j = 0$ then stop; else

(4) let $u := u + s_{j-1}$; $v := v + s_{j-1}$; $X_j := \mathrm{diag}(I_u, \hat{X}_j, I_v)$; compute $X := XX_j$; $A := X^T A X$; $M := X^T M X$;

(5) let $n_{j+2} := n_j - s_j - s_{j+1}$; define A_{j+2} and M_{j+2} by deleting the first s_{j+1} columns and rows and the last s_j columns and rows of A_j and M_j respectively; $j := j + 2$; GOTO (2).

Note that this algorithm stops when M_{j+2} has full rank and reduces $A - \lambda M$ to the following form shown in Figure 1 (we reuse some of the names of the blocks), where

(A1) M_{j+2} has full rank (thus $s_{j+2} = 0$);

(A2) Φ_i and Θ_i are nonzero diagonal matrices with ranks a_i and s_{i+1} respectively and satisfying $a_i + s_{i+1} = s_i$ for $i = 1,3,\dots,j$;

(A3) $[\hat{K}_i \,|\, \hat{G}_{i,1} \,|\, \hat{G}_{i,2}]$ have full row rank s_{i+1} for $i = 1,3,\dots,j$;

(A4) $\hat{G}_{i,2}$ has full column rank s_{i+2} for $i = 1,3,\dots,j-2$.

From (A1)–(A4) it follows that the s_i form a decreasing sequence and thus

$$s_i - s_{i+1} = a_i \geqslant 0 \quad \text{for} \quad i = 1,2,3,\dots,j. \tag{2.6}$$

In Lemmas 2.1 and 2.2 and Proposition 2.4 we will prove that the forms (A1)–(A4) and (2.6) indeed yield the Jordan structure of $A - \lambda M$ at infinity.

LEMMA 2.1. *The pencil in Figure 1 which is obtained from Algorithm 2.1 by stopping at stage $j+2$ is strictly equivalent to the pencil*

$$\mathrm{diag}\{A_f - \lambda M_f, A_\infty - \lambda M_\infty\}, \qquad \textit{where} \quad A_f \equiv A_{j+2} \ \textit{and} \ M_f \equiv M_{j+2}. \tag{2.7}$$

Here A_∞ and M_∞ ($\in \mathbb{R}^{h \times h}$ with $h = u + v$) have the staircase form shown in Figure 2 (the general case will be sufficiently illustrated by considering the case $j = 5$), where i_k, j_k indicate respectively the corresponding row and

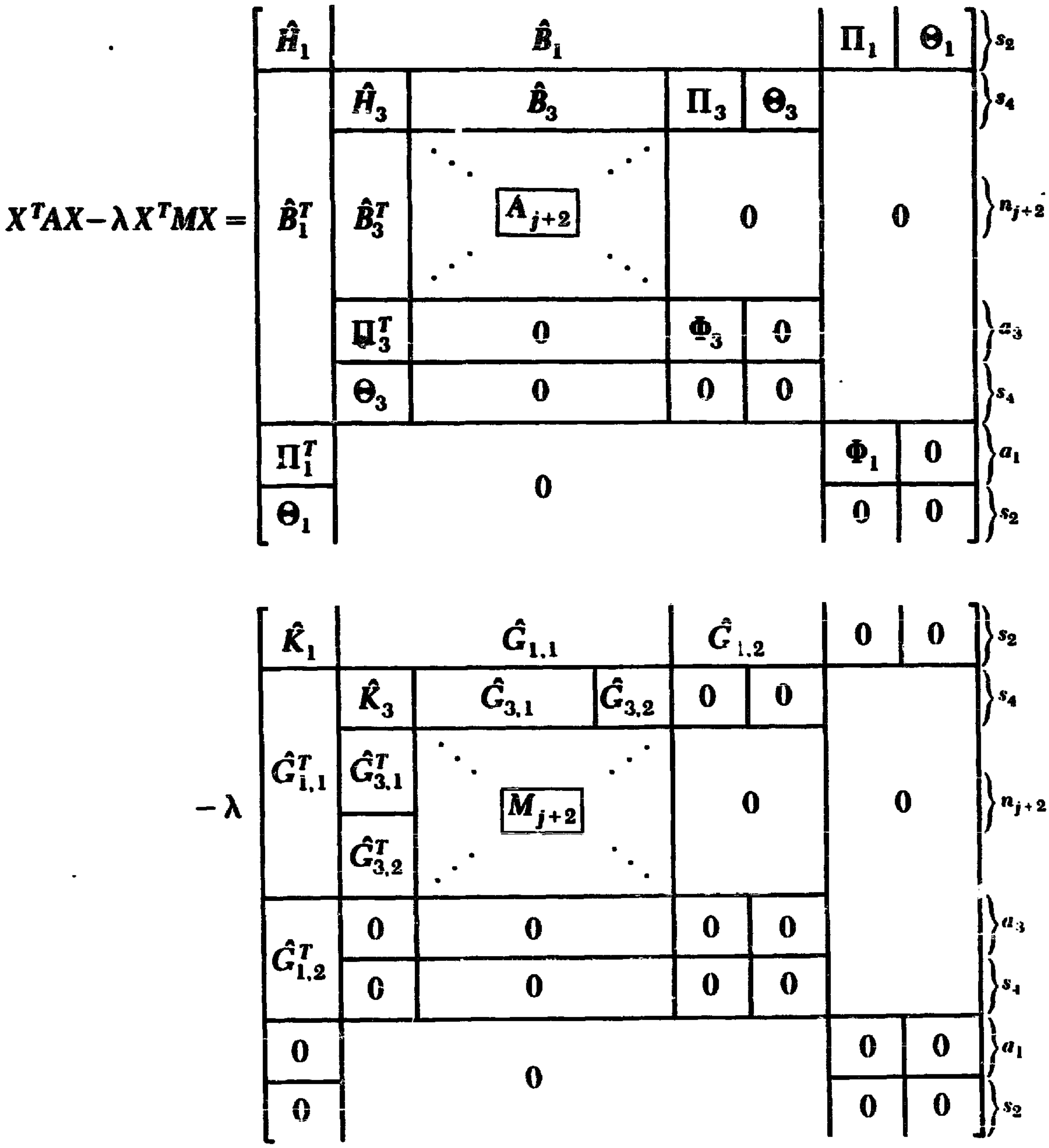

FIG. 1.

column of M_∞; *and*

(B1) $M_\infty(1,i_1;1,j_5)$ *and* $M_\infty(i_1,i_2;j_1,j_4)$ *have full row rank;*

(B2) $M_\infty(1,i_5;1,j_1)$ *and* $M_\infty(i_1,i_4;j_1,j_2)$ *have full column rank;*

(B3) $M_\infty(1,i_1;j_4,j_5)=M_\infty^T(i_4,i_5;1,j_1)$ *and* $M_\infty(i_1,i_2;j_3,j_4)=M_\infty^T(i_3,i_4;j_1,j_2)$ *have full column rank;*

(B4) $M_\infty(i_2,i_3;j_2,j_3)=M_\infty^T(i_2,i_3;j_2,j_3)$ *has full rank, and* Φ_i,Θ_i $(i=1,3,5)$ *are nonzero diagonal matrices.*

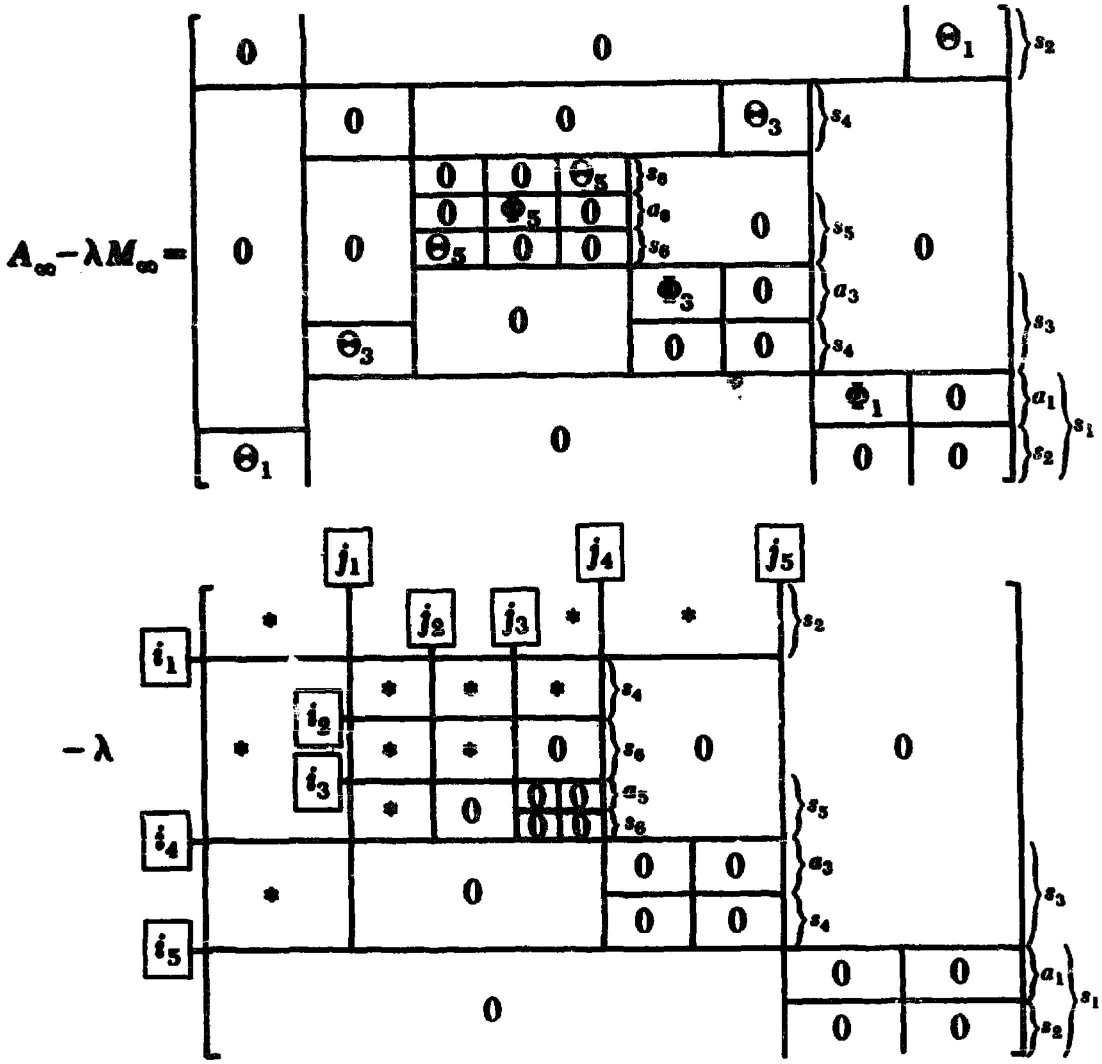

FIG. 2.

Proof. We prove this inductively by constructing the transformations that zero all blocks on the left of M_f and A_f in Figure 1 respectively. Since M_f has full rank, there exists a column transformation that uses this block to zero out the first block on the left of M_f. Afterwards the modified nonzero block left on A_f is eliminated by a row transformation using Θ_j as pivot. We do this by induction, using M_f and $\Theta_j, \Theta_{j-2}, \ldots, \Theta_1$ as pivots. Performing some suitable permutations, we then have the following equivalent pencils:

$$A - \lambda M \sim \begin{bmatrix} A_f & 0 \\ * & \tilde{A}_\infty \end{bmatrix} \}\, h - \lambda \begin{bmatrix} M_f & 0 \\ * & M_\infty \end{bmatrix} \}\, h, \tag{2.8}$$

where $\tilde{A}_\infty$ has the same form as A_∞ except for some staircases above and to the left of Φ_j and Θ_i $(i = 1, 3, \ldots, j)$.

Now we zero out all blocks respectively above and to the left of Φ_j and Θ_i ($i = 1,3,\ldots, j$), using $\Theta_j,\ldots,\Theta_1$ as row and column pivots, and then obtain that Figure 1 is equivalent to Figure 2. The conditions (B1)–(B4) are clearly obtained from Algorithm 2.1. Further, the last h columns of the right transformation matrix in (2.8) or (2.7) are a basis for the invariant subspace corresponding to the infinite eigenvalue of $A - \lambda M$. ∎

LEMMA 2.2. *The pencil $A_\infty - \lambda M_\infty$ in Lemma 2.1 is strictly equivalent to the pencil $H_h - \lambda \tilde{M}_\infty$, where $\tilde{M}_\infty$ and H_h have the staircase forms shown for the case $j = 5$ in Figure 3 (where every $*$ is a nonsingular square matrix with the corresponding size and a blank denotes the zero matrix) and Figure 4.*

Proof. Since the pencil $A_\infty - \lambda M_\infty$ is symmetric in block form (see Figure 2), we only need to consider the blocks in the upper triangle; those in the lower triangle can be treated in the same way. We prove this inductively by constructing column or row transformations and SVD that zero out the nonzero blocks of M_∞ in Figure 2 to the form of Figure 3, using appropriate nonsingular matrices as pivots.

We begin with the central submatrix of M_∞, shown for the case $j = 3$ in Figure 5. From the full column and row rank conditions (B1)–(B4) we zero out the nonzero blocks in above numerical order from 1 to 6 (see Figure 5), where

$$\overleftarrow{(i)} \quad \text{and} \quad \uparrow (j)$$

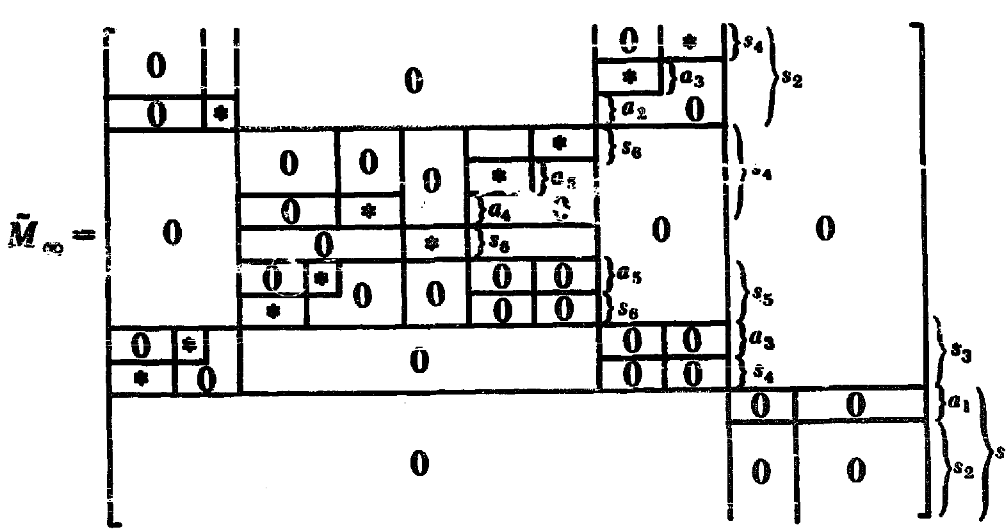

FIG. 3.

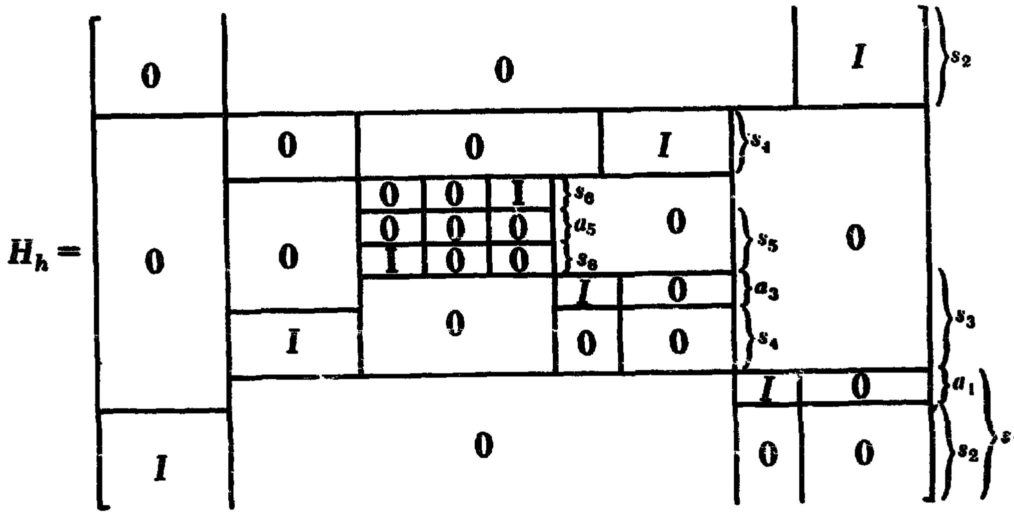

FIG. 4.

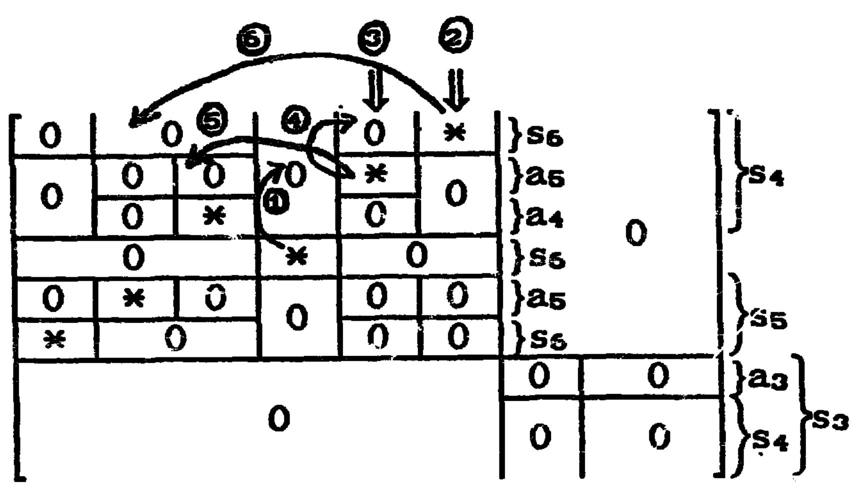

FIG. 5.

denote column and row transformations respectively, using the nonsingular matrices $*$ as pivots, $\Downarrow$ (i) denotes singular value decomposition of an $m \times n$ matrix with $m > n$. The modified nonzero blocks above Θ_3, Θ_5, Φ_5, and Θ_5 in Figure 2 are eliminated immediately by column and row transformations using Θ_3 as pivot (see Figure 6).

We now consider the submatrix of M_∞ as for the case $j = 5$, and so on; we perform the eliminations in the above order (for the case $j = 3$) in an analogous fashion. We conclude that $A_\infty - \lambda M_\infty \sim \hat{A}_\infty - \lambda \tilde{M}'_\infty$, where $\hat{A}_\infty$ has the same block form as A_∞, and $\tilde{M}'_\infty$ has the form of Figure 3. To unitize the block-diagonal $\hat{\Theta}_1, \hat{\Theta}_3, \ldots, \hat{\Theta}_j$ in $\hat{A}_\infty$ by multiplying by $\hat{\Theta}_1^{-1}, \ldots, \hat{\Theta}_j^{-1}$ on

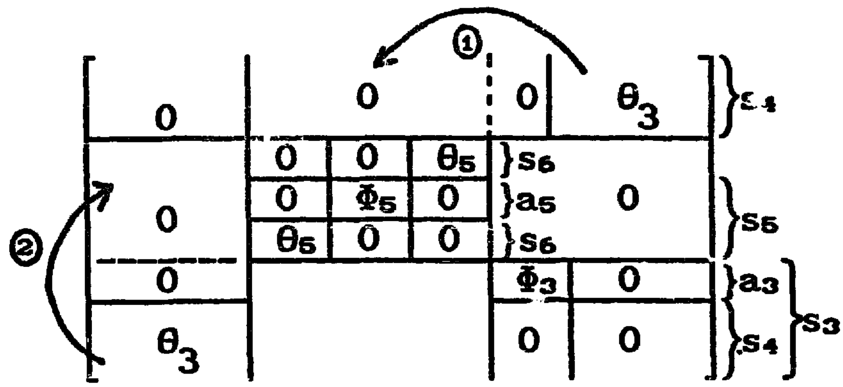

FIG. 6.

the left, and $\hat{\Phi}_j, \hat{\Theta}_j, \hat{\Phi}_{j-2}, \hat{\Theta}_{j-2}, \ldots, \hat{\Phi}_1, \hat{\Theta}_1$ by multiplying by $\hat{\Phi}_j^{-1}, \hat{\Theta}_j^{-1}, \ldots, \hat{\Phi}_1^{-1}, \Theta_1^{-1}$ on the right, we then have $A_\infty - \lambda M_\infty \sim H_h - \lambda \tilde{M}_\infty$. ∎

LEMMA 2.3. *The pencil $H_h - \lambda \tilde{M}_\infty$ $(\in \mathbf{R}^{h \times h})$ in Lemma 2.2 is strictly equivalent to the pencil*

$$H_h - \lambda \hat{M}_\infty \equiv \begin{bmatrix} 0_{s_2} & & & & & & & K_2 \\ & \ddots & & & & & \iddots & \\ & & 0_{s_{j-1}} & 0 & 0 & K_{j-1} & & \\ & & 0 & 0_{s_{j+1}} & K_{j+1} & 0 & & \\ & & 0 & K_{j+1}^T & I_j' & 0 & & \\ & & K_{j-1}^T & 0 & 0 & I_{j-2}' & & \\ & \iddots & & & & & \ddots & \\ K_2 & & & & & & & I_1' \end{bmatrix}$$

$$-\lambda \begin{bmatrix} \hat{I}_2 & & & & & & G_2 & 0 \\ & \ddots & & & & \iddots & & \\ & & \hat{I}_{j-1} & 0 & G_{j-1} & & & \\ & & 0 & \hat{I}_{j+1} & 0 & & & \\ & & G_{j-1}^T & 0 & 0_{s_j} & & & \\ & \iddots & & & & \ddots & & \\ G_2^T & & & & & & 0_{s_3} & \\ 0 & & & & & & & 0_{s_1} \end{bmatrix}, \quad (2.9)$$

where

$$K_i = \underbrace{[0 \mid I]}_{s_{i-1}}\}\, s_i \qquad (i = 2,4,\ldots,j+1),$$

$$I_i' = \underbrace{\begin{bmatrix} I_{a_i} & 0 \\ 0 & 0 \end{bmatrix}}_{s_i}\Bigg\}\, s_i \qquad (i = 1,3,\ldots,j), \tag{2.10}$$

and

$$G_i = \left.\begin{array}{c} \overbrace{\hphantom{\begin{matrix} 0 & I \end{matrix}}}^{s_{i+1}} \\ \left[\begin{array}{cc} 0 & I \\ I & 0 \\ \hline \multicolumn{2}{c}{0} \end{array}\right] \begin{array}{l} \}\, s_{i+2} \\ \\ \}\, a_{i+1} \end{array} \end{array}\right\}\, s_i \qquad (i = 2,4,\ldots,j-1),$$

$$\hat{I}_i = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & I_{a_i} \end{bmatrix}}_{s_i}\Bigg\}\, s_i \qquad (i = 2,4,\ldots,j+1). \tag{2.11}$$

Proof. $\tilde{M}_\infty$ in Figure 3 can be written in the form

$$\begin{bmatrix} \tilde{I}_2 & & & & & & \tilde{G}_2 & 0 \\ & \ddots & & & & \iddots & & \\ & & \tilde{I}_{j-1} & 0 & \tilde{G}_{j-1} & & & \\ & & 0 & \tilde{I}_{j+1} & 0 & & & \\ & & \tilde{G}_{j-1}^T & 0 & 0_{s_j} & & & \\ & \iddots & & & & \ddots & & \\ \tilde{G}_2^T & & & & & & 0_{s_3} & \\ 0 & & & & & & & 0_{s_1} \end{bmatrix}$$

with

$$\tilde{G}_i = \left.\begin{array}{c} \overbrace{\hphantom{0 \quad\quad *}}^{s_{i+1}} \\ \left[\begin{array}{cc} 0 & * \\ * & 0 \\ \hline \multicolumn{2}{c}{0} \end{array}\right] \end{array} \begin{array}{l} \\ \}\, s_{i+2} \\ \}\, a_{i+1} \\ \\ \end{array}\right\} s_i \qquad (i=2,4,\ldots,j-1)$$

and

$$\tilde{I}_i = \left.\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}}_{s_i} \begin{array}{l} \\ \}\, a_i \end{array}\right\} s_i \qquad (i=2,4,\ldots,j+1).$$

There exists a sequence of row transformations which can normalize $\tilde{I}_{j+1}$, $\tilde{I}_{j-1}$, and $\tilde{G}_{j-1}$ to $\hat{I}_{j+1}$, $\hat{I}_{j-1}$, and G_{j-1} respectively. Though the transformations affect the matrices I_j', K_{j+1}, and K_{j-1}, they can be immediately normalized to the form (2.10) by using some suitable column transformations. The matrices $\tilde{G}_{j-1}^T$, K_{j+1}^T, and K_{j-1}^T in the lower triangles of $\hat{M}_\infty$ and H_h can also be normalized by the transposes of the same transformations as above. This process of normalization can be continued throughout the whole matrix for $i=j+1, j-1,\ldots,2$ and yields the result (2.9). ∎

Proposition 2.4. *The indices $\{s_i\}$ given by Algorithm 2.1 completely determine the structure at ∞ of the pencil $A-\lambda M$: $A-\lambda M$ has $a_i(=s_i-s_{i+1})$ elementary divisors $(1/\lambda)^i$ $(i=1,\ldots,j+1)$.*

Proof. By Lemmas 2.1, 2.2, and 2.3 we have that the pencil $A-\lambda M$ is strictly equivalent to the pencil $H_h-\lambda\hat{M}_\infty$ as in (2.9).

By Algorithm 2.1 and (2.6) we obtain that

$$s_1 \geqslant s_2 \geqslant \cdots \geqslant s_{j+1} \geqslant s_{j+2} = 0 \tag{2.12a}$$

and

$$s_i - s_{i+1} = a_i \qquad \text{for} \quad i=1,2,\ldots,j+1. \tag{2.12b}$$

From (2.12) and the relations (2.10), (2.11) it follows that

$$h=\sum_{i=1}^{j+1} s_i=\sum_{i=1}^{j+1}\left(\sum_{l=i}^{j+1} a_l\right)=\sum_{l=1}^{j+1} l a_l. \tag{2.13}$$

It is easy to see that by performing some suitable row and column permutations the pencil $H_h-\lambda\tilde{M}_\infty$ is equivalent to the pencil shown in Figure 7. Again performing a separation of structure elements by permutations only,

$$\begin{bmatrix} \left.\boxed{\begin{matrix} 0 & & I_{a_{j+1}} \\ & \cdot^{\cdot^{\cdot}} & \\ I_{a_{j+1}} & & 0 \end{matrix}}\right\}(j+1)a_{j+1} & & & 0 \\ & \ddots & & \\ & & \boxed{\begin{matrix} 0 & I_{a_2} \\ I_{a_2} & 0 \end{matrix}} & \\ 0 & & & \boxed{I_{a_1}} \end{bmatrix}$$

$$-\lambda \begin{bmatrix} \left.\boxed{\begin{matrix} 0 & & I_{a_{j+1}} & & 0_{a_{j+1}} \\ & \cdot^{\cdot^{\cdot}} & & & \\ I_{a_{j+1}} & & \cdot^{\cdot^{\cdot}} & & \\ 0_{a_{j+1}} & & & & 0 \end{matrix}}\right\}(j+1)a_{j+1} & & & 0 \\ & \ddots & & \\ & & \boxed{\begin{matrix} I_{a_2} & 0 \\ 0 & 0_{a_2} \end{matrix}} & \\ 0 & & & \boxed{0_{a_1}} \end{bmatrix}$$

FIG. 7.

the pencil in Figure 7 can be brought to the canonical form

$$\bigoplus_{q=1}^{j+1}\left(\bigoplus_{p=1}^{a_q} N_q\right), \tag{2.14}$$

where

$$N_q := \begin{bmatrix} 0 & & & \lambda & 1 \\ & & \cdot & \cdot & \\ & \cdot & \cdot & & \\ \lambda & \cdot & & & \\ 1 & & & & 0 \end{bmatrix}_{q\times q}$$

as in (1.2). We then obtain the desired result. ■

REMARK 2.5(1). If α is a real eigenvalue, in order to determine an elementary divisor of the finite eigenvalue α, we need only replace the roles of A and M in Algorithm 2.1 by M and $A-\alpha M$ respectively. When α is a complex conjugate eigenvalue, it is required to revise some substeps in step 1 of Algorithm 2.1(2) as follows: Let A_1 and M_1 be two symmetric complex matrices.

Step 1′. (a′) Compute the singular value decomposition of M_1 [as in (1.5)]. The matrices Q_1 and Λ_1 in (2.0) are unitary and positive diagonal respectively.

(b′) Compute the singular value decomposition of H_1 [as in (1.5)]. The matrices $\tilde{P}_1$ and Φ_1 in (2.2) are unitary and positive diagonal respectively.

(c′) Compute the singular value decomposition of $\tilde{G}_1$ [as in (1.4)],

$$\bar{\hat{U}}_1^T\tilde{G}_1\hat{V}_1 = \begin{bmatrix}\Theta_1\\ 0\end{bmatrix}\}s_2,$$

and replace $\hat{U}^T$ by $\bar{\hat{U}}^T$ throughout substep (c).

The other procedures and notation of step 1′, which we have not mentioned, are the same as step 1.

We then get a similar algorithm, say Algorithm 2.1′.

Indeed, at the beginning the finite eigenvalues are not known. Thus one can perform Algorithm 2.1 to cancel all infinite eigenvalues of the pencil $A-\lambda M$ and deflate it to a smaller regular pencil $A_f-\lambda M_f$ (see Lemma

2.1), which contains only finite zeros, and then one can use the well-known QZ [9] or HR [1, 2] algorithms to compute the finite zeros α (say). In fact, if infinite elementary divisors of degree higher than one occur in the pencil, it can be troublesome for the QZ algorithm; hence we must first get rid of infinite (or very large) eigenvalues and then use Algorithm 2.1 or 2.1′ on the pencil $M-\lambda(A-\alpha M)$ to compute a Kronecker basis (i.e. a basis of the invariant subspace associated to α) and further to determine the Kronecker canonical form corresponding to α. This is done by inspecting the rank deficiency of $A-\lambda M$ (for ∞) and $M-\lambda(A-\alpha M)$ (for α).

REMARK 2.5(2). From Lemma 2.1 one can transform $A-\lambda M$ to the form (2.8), yielding the infinite elementary divisors and a deflated pencil $A_f-\lambda M_f$ with only finite eigenvalues. At this stage the last h columns of right transformation matrix are recognizable as a Kronecker basis of the invariant subspace of the pencil $\tilde{A}_\infty-\lambda M_\infty$ as in (2.8), and only stable transformations—including products of the factors Φ_i^{-1} $(i=1,\dots,j)$ in Algorithm 2.1 and $1/\sqrt{|d_i|}$ $(i=1,\dots,n-h)$—have been used, where d_i is the ith eigenvalue of M_f.

In fact, in Lemma 2.1 it is not necessary to compute the inverse of M_f to zero out the blocks on the left of M_f. We have the following method: Let η be a given criterion for negligibility. There exists an orthogonal matrix Q such that $Q^TM_fQ=\operatorname{diag}\{d_1,\dots,d_{n-h}\}\equiv D$ and $|d_i|>\eta$; it follows that $\|D^{-1/2}\|<1/\sqrt{\eta}$. We have $|D|^{-1/2}Q^TM_fQ|D|^{-1/2}\equiv J$ (a diagonal matrix with ±1 on the diagonal). Then use this matrix to zero out all blocks on the left of it (the large factors $1/\sqrt{|d_i|}$ in these blocks can be canceled by multiplying by $\sqrt{|d_i|}$ on the right). Thus we have at most the factor $1/\sqrt{\eta}$ affecting the norm of the right transformation matrix, instead of $1/\eta$. From step 1(b) of Algorithm 2.1 and (1.3) we have $\eta<\sigma_{\min}(\Phi_i)$ (the smallest singular value of Φ_i) for $i=1,\dots,j$; it follows that $\|\Phi_i^{-1}\|<1/\eta$. On the other hand, Φ_i are very small compared with the whole matrix; hence, though Φ_i^{-1} affects the stability of transformed matrices, the norms of these matrices do not grow large quickly. Therefore this procedure, used to compute a Kronecker basis associated to $\tilde{A}_\infty-\lambda M_\infty$, is still fairly stable.

We have a similar discussion for the case of complex matrices A and M. If the right transformation matrices that bring $A-\lambda M$ to its Kronecker canonical form are needed, one must zero out some staircase forms over the subdiagonal and perform transformations as in Lemmas 2.2 and 2.3. These are Gaussian-type eliminations without pivoting. The computation of this information may thus be very unstable.

REMARK 2.5(3). If, in spite of the symmetry, we use Van Dooren's algorithm [10] on the symmetric pencil $A-\lambda M$ for computing the Kronecker structure, then we get only one piece of information s_i, i.e. the nullity of the current matrix in step i (see [10, Algorithm 3.1]). In order to obtain the number of Kronecker chains of size i $(a_i \equiv s_i - s_{i+1})$, it is necessary to compute the nullity s_{i+1} of the next current matrix in step $i+1$. The dimensions of the current matrices in steps i and $i+1$ [10, Algorithm 3.1] are n_i and $n_i - s_i$ respectively, where $n_i = n - s_1 - \cdots - s_{i-1}$. Now if we use Algorithm 2.1 or 2.1′ on $A - \lambda M$ to determine the number of Kronecker chains of size i $(a_i \equiv s_i - s_{i+1})$, we need to compute the nullity s_i of M_i [as in (2.0)] and the nullity s_{i+1} of H_i [as in (2.2)]. We know that M_i has dimension n_i $(n_i = n - s_1 - \cdots - s_{i-1})$, but H_i has dimension s_i; if $s_i \ll n$ for all i, then on using Algorithm 2.1 or 2.1′ instead of Van Dooren's algorithm [10] one almost halves the operations in one step, and the symmetry is also preserved.

3. SYMMETRIC SINGULAR PENCILS

The structure of singular pencils is more complex than that of regular pencils because of the occurrence of the Kronecker blocks as in (1.2). In this section we demonstrate how to separate the finite regular pencil $A_f - \lambda M_f$ and the singularity $\operatorname{diag}\{A_\eta - \eta M_\eta, A_\infty - \lambda M_\infty, A_\epsilon - M_\epsilon\}$ with an algorithm similar to Algorithm 2.1. Thereby $A_\epsilon - \lambda M_\epsilon$ and $A_\eta^T - M_\eta^T$ become equivalent to $\oplus_{i=1}^{p} L_{\epsilon_i}$, and we have $A_\infty - \lambda M_\infty \sim \oplus_{i=1}^{s} N_i$ and $A_f - \lambda M_f \sim \oplus_{i=1}^{r} Z_i$ as in (1.2). When the pencil $A - \lambda M$ is singular, the matrix $\tilde{G}_1$ in (2.3) no longer has full column rank, but using some appropriate partitions and transformations similar to step 1(a)–(d) of Algorithm 2.1(2), one can also obtain the following result like (2.4) (here we use the same notation as in step 1 and set $A_1 := A$, $B_1 := B$, and $n_1 := n$):

$$X_1^T(A_1 - \lambda M_1)X_1 = \begin{bmatrix} \hat{H}_1 & \hat{B}_1 & \Pi_1 & \Theta_1 & 0 \\ \hat{B}_1^T & A_3 & 0 & 0 & 0 \\ \Pi_1^T & 0 & \Phi_1 & 0 & 0 \\ \Theta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \}\, s_2 \\ \}\, n_1 - s_1 - s_2 \\ \left.\begin{array}{l} \left.\begin{array}{l} \}\, d_1 \\ \}\, s_2 \end{array}\right\} r_1 \\ \}\, e_1 \end{array}\right\} s_1 \end{array}$$

$$-\lambda \begin{bmatrix} K_1 & \hat{G}_1 & 0 & 0 & 0 \\ \hat{G}_1^T & M_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \}\, s_2 \\ \}\, n_1 - s_1 - s_2 \\ \left.\begin{array}{l} \left.\begin{array}{l} \}\, d_1 \\ \}\, s_2 \end{array}\right\} r_1 \\ \}\, e_1 \end{array}\right\} s_1 \end{array} \tag{3.1}$$

where X_1 is the product of the transformation matrices, $s_1 = e_1 + r_1$, and $\operatorname{rank}(\tilde{G}_1) = s_2$.

We repeat this procedure (3.1) until the matrix M_{j+2} $(j = 1, 3, \ldots)$ has full rank, as described in the following algorithm.

ALGORITHM 3.1.

(1) $j := 1$; $A_1 := A$; $M_1 := M$; $n_1 := n$; $X := I_n$; $Y^T := I_n$; $u_1 := u_2 := v_1 := v_2 := 0$;

(2) repeat the above procedure (3.1) on the $n_j \times n_j$ matrix pencil $A_j - \lambda M_j$; compute the transformation matrix $\hat{X}_j := Q_j \hat{P}_j U_j V_j F_j$; let $s_j :=$ nullity of M_j; $d_j :=$ rank of Φ_j; $s_{j+1} :=$ rank of Θ_j; $r_j := d_j + s_{j+1}$; $e_{(j+1)/2} := s_j - r_j$; $d_{j-1} := s_{j-1} - s_j$ $(d_0 = 0)$;

(3) if $s_j := 0$ then stop; else

(4) let $u_1 := u_1 + s_{j-1}$; $v_1 := v_1 + s_{j-2}$; $u_2 := u_2 + s_{j-1} + e_{(j-1)/2}$; $v_2 := v_2 + r_{j-2}$ $(e_0 = r_{-1} = s_{-1} = s_0 = 0)$; $X_j := \operatorname{diag}(I_{u_1}, \hat{X}_j, I_{v_1})$; compute $X := XX_j$; $A := X^T AX$; $M := X^T MX$;

(5) let $n_{j+2} := n_j - s_j - s_{j+1}$; define A_{j+2} and M_{j+2} by deleting the first s_{j+1} columns and s_{j+1} rows and the last s_j columns and s_j rows of A_j and M_j respectively; $j := j + 2$; GOTO (2).

This algorithm stops when M_{j+2} has full rank. As in Lemma 2.1, we can also use this matrix to zero out the nonzero blocks on the left of it, and eliminate the modified nonzero blocks left on A_{j+2} by some row transformations using Θ_j as pivot. Then we have

LEMMA 3.1. *The pencil $A - \lambda M$ is strictly equivalent to the pencil*

$$\begin{bmatrix} A_f - \lambda M_f & 0 \\ * & A_x - \lambda M_x \end{bmatrix} \}h, \tag{3.2}$$

where $A_f \equiv A_{j+2}$, $M_f \equiv M_{j+2}$ and $A_x - \lambda M_x$ has the staircase form shown in Figure 8 (the general case will be sufficiently illustrated by considering the case $j = 3$), and where

(C1) *$M_x(1, i_1; 1, j_4)$ and $M_x^T(1, i_4; 1, j_1)$ have full row rank;*

(C2) *$M_x(1, i_1, j_3, j_4) = M_x^T(i_3, i_4; 1, j_1)$ has full column rank;*

(C3) *$M_x(i_1, i_2; j_1, j_2) = M_x^T(i_1, i_2; j_1, j_2)$ has full rank, and Θ_i, Φ_i $(i = 1, 3)$ are nonzero diagonal matrices.*

Proof. The proof is similar to that of Lemma 2.1; the conditions (C1)–(C3) are obtained immediately from Algorithm 3.1. ∎

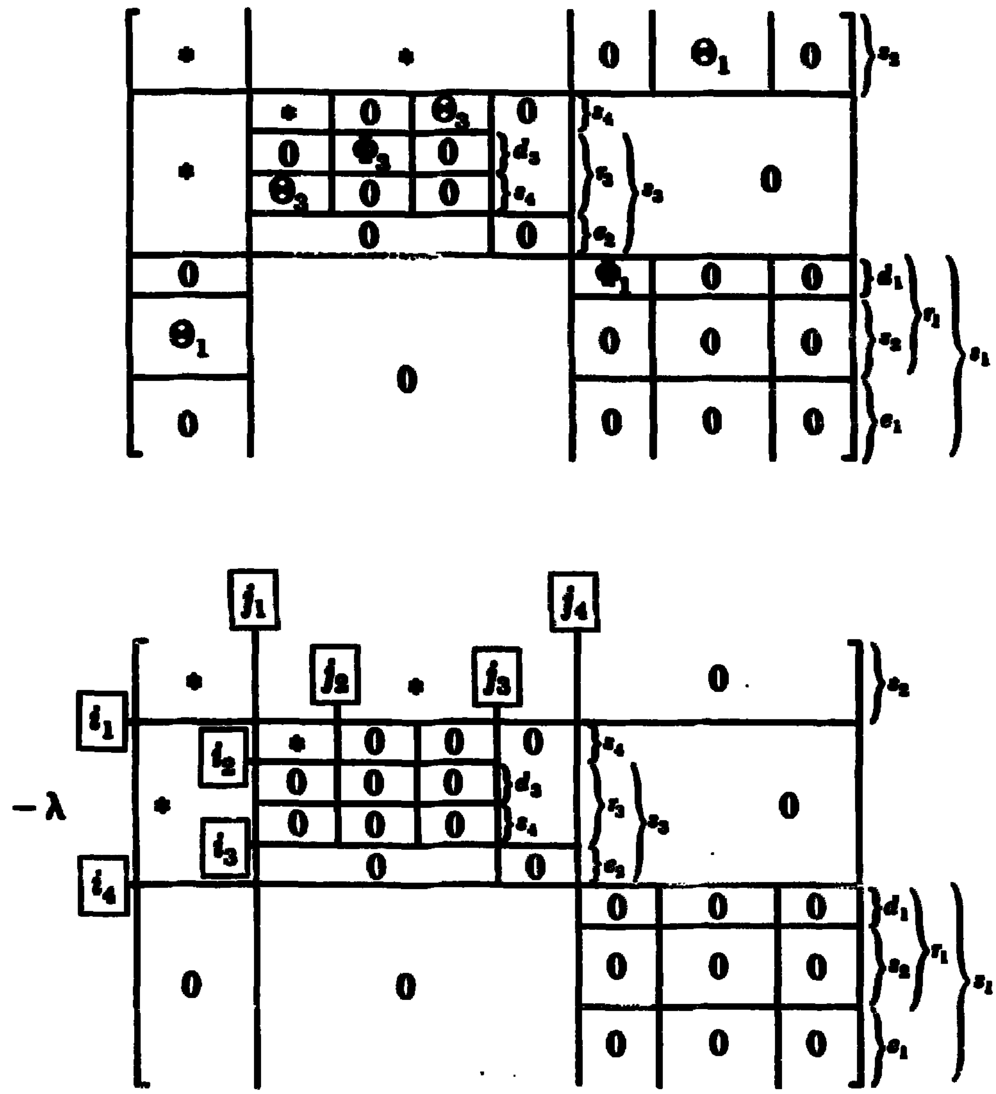

FIG. 8.

LEMMA 3.2. *The pencil $A_x - \lambda M_x$ in Lemma 3.1 is strictly equivalent to the pencil $H_h - \lambda \tilde{M}_x$, where H_h has the same form of A_x in Figure 8, except that all Θ_i, Φ_i $(i = 1,3,\ldots, j)$ and the staircase are replaced by identity and zero matrices respectively, and $\tilde{M}_x$ has the staircase form shown for the case $j = 5$ in Figure 9, where every * is a nonsingular square matrix with the appropriate size.*

Proof. Using the conditions (C1)–(C3) it is not difficult to see that we can also perform a procedure like that in Figures 5 and 6 to reduce M_x to the form in Figure 9. We omit the details. ■

LEMMA 3.3. *Suppose $j=2k-1$ and Algorithm 3.1 stops at step $j+2$. Then the pencil $H_h-\lambda \tilde{M}_z$ in Lemma 3.2 is strictly equivalent to the pencil*

$$A_d-\lambda M_d \tag{3.3}$$

with the form

$$A_d=\begin{bmatrix} 0_{s_2} & & & & & & K_2 \\ & \ddots & & & & \iddots & \\ & & 0_{s_{j-1}} & 0 & 0 & K_{j-1} & \\ & & 0 & 0_{s_{j+1}} & K_{j+1} & 0 & \\ & & 0 & K_{j+1}^T & I_j' & 0 & \\ & & K_{j-1}^T & 0 & 0 & I_{j-2}' & \\ & \iddots & & & & & \ddots \\ K_2^T & & & & & & I_1' \end{bmatrix}, \tag{3.4}$$

$$M_d=\begin{bmatrix} \hat{I}_2 & & & & & & G_2 & 0_{s_2,s_1} \\ & \ddots & & & & \iddots & & \\ & & \hat{I}_{j-1} & 0 & G_{j-1} & & & \\ & & 0 & \hat{I}_{j+1} & 0 & & & \\ & & G_{j-1}^T & 0 & 0_{s_j} & & & \\ & \iddots & & & & \ddots & & \\ G_2^T & & & & & & 0_{s_3} & \\ 0_{s_2,s_1} & & & & & & & 0_{s_1} \end{bmatrix}, \tag{3.5}$$

where

$$K_i=\overbrace{\underbrace{[0 \mid}_{d_{i-1}} \; I_{s_i} \; \underbrace{\mid 0]}_{e_{i/2}}}^{s_{i-1}} \Big\} s_i \qquad (i=2,4,\ldots,j+1),$$

$$\tag{3.6}$$

$$I_i'=\left.\begin{array}{l}\begin{bmatrix} I_{d_i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{array}{l} \\ \}\, s_{i+1} \\ \}\, e_{(i+1)/2} \end{array}\end{array}\right\} s_i \qquad (i=1,3,\ldots,j)$$

$\tilde{M}_x =$

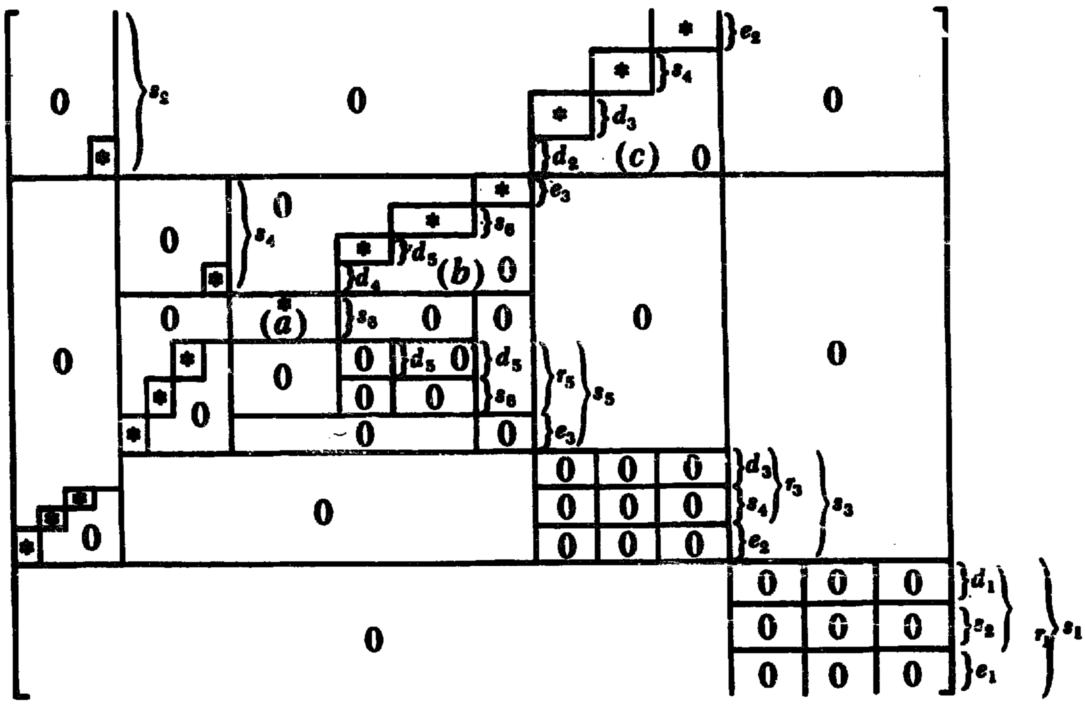

FIG. 9.

and

$$G_i = \left.\begin{array}{c}\overbrace{\hphantom{I_{d_{i+1}} \quad I_{s_{i+2}} \quad I_{e_i/2}}}^{s_{i+1}}\\ \left[\begin{array}{ccc} & & I_{e_i/2}\\ & I_{s_{i+2}} & \\ I_{d_{i+1}} & & \\ \hline & 0 & \end{array}\right]\end{array}\right\} s_i \qquad (i = 2, 4, \ldots, j-1), \tag{3.7}$$

$$\hat{I}_i = \left.\begin{bmatrix} 0 & 0 \\ 0 & I_{d_i} \end{bmatrix}\right\} s_i \qquad (i = 2, 4, \ldots, j+1),$$

where $d_i := s_i - s_{i+1}$.

Proof. Since every $*$ in Figure 9 is nonsingular, we first normalize the nonsingular matrices on the upper right corner—for example those in positions (a), (b), and (c)—to the identity and preserve H_h. The lower left corner can be treated in the same way. ∎

Proposition 3.4. *Let $j = 2k-1$. The indices $\{e_i \mid i = 1,\dots,k\}$ and $\{d_i \mid i = 1,\dots,j+1\}$ determine the Kronecker structure of the pencil $A - \lambda M$:*

(i) *there are d_i infinite elementary divisors of degree i ($i = 1,\dots,j+1$);*
(ii) *there are e_i Kronecker blocks L_{i-1} of size $i-1$ ($i = 1,\dots,k$);*
(iii) *there are e_i Kronecker blocks L_{i-1}^T of size $i-1$ ($i = 1,\dots,k$).*

Proof. By Algorithm 3.1 and (3.4)–(3.7) we have the following relations:

$$\begin{aligned}
&s_1 - r_1 = e_1, \quad s_3 - r_3 = e_2, \dots, \qquad s_j - r_j = e_k,\\
&r_1 - s_2 = d_1, \quad r_3 - s_4 = d_3, \dots, \quad r_j - s_{j+1} = d_j, \\
&s_2 - s_3 = d_2, \quad s_4 - s_5 = d_4, \dots, \qquad s_{j+1}^{-0} = d_{j+1} \qquad (j = 2k-1).
\end{aligned} \tag{3.8}$$

It follows that

$$\begin{aligned}
h &= \sum_{i=1}^{j+1} s_i = \sum_{i=1}^{j+1} \left(\sum_{l=i}^{j+1} d_l \right) + \sum_{l=1}^{(j+1)/2} (2l-1)e_l \\
&= \sum_{l=i}^{j+1} l d_l + \sum_{l=1}^{j+1/2} (2l-1)e_l .
\end{aligned}$$

Now, it is easy to see that by performing some suitable permutations the pencil $A_x - \lambda M_x$ is equivalent to the pencil $(A_y - \lambda M_y) \oplus (A_\infty - \lambda M_\infty)$, where the factors are shown in Figures 10 and 11. Again performing a separation of structure elements by permutations only the pencil $(A_y - \lambda M_y) \oplus (A_\infty - \lambda M_\infty)$ can be brought to the canonical form

$$\left(\bigoplus_{q=1}^{k} \bigoplus_{p=1}^{e_q} \begin{bmatrix} 0 & L_{q-1}^T \\ L_{q-1} & 0 \end{bmatrix} \right) \oplus \left(\bigoplus_{q=1}^{j+1} \bigoplus_{p=1}^{d_q} N_q \right), \tag{3.9}$$

where L_{q-1} and N_q are defined in (1.2). ∎

Corollary 3.5. *One can construct unitary transformations that put the pencil $A_x - \lambda M_x$ in Lemma 3.1 in the following form (P and Q are unitary*

$$
A_\infty - \lambda M_\infty = \left[\begin{array}{ccc}
\begin{array}{|ccc|}
0 & & I_{d_{j+1}} \\
 & \cdot^{\cdot^{\cdot}} & \\
I_{d_{j+1}} & & 0 \\ \hline
\end{array} & & 0 \\
 & \ddots & \\
0 & & \begin{array}{|cc|c}
0 & I_{d_2} & \\
I_{d_2} & 0 & \\ \hline
 & & \multicolumn{1}{|c}{I_{d_1}}
\end{array}
\end{array}\right]
$$

$$
- \lambda \left[\begin{array}{ccc}
\begin{array}{|ccc|}
 & \cdot^{\cdot^{\cdot}}\ I & 0_{d_{j+1}} \\
I & \cdot^{\cdot^{\cdot}} & \\
0_{d_{j+1}} & & 0 \\ \hline
\end{array} & & 0 \\
 & \ddots & \\
0 & & \begin{array}{|cc|c}
I_{d_2} & 0 & \\
0 & 0 & \\ \hline
 & & \multicolumn{1}{|c}{0_{d_1}}
\end{array}
\end{array}\right]
$$

FIG. 10.

matrices):

$$
P(A_x - \lambda M_x)Q = \left[\begin{array}{c|c|c}
A_\eta - \lambda M_\eta & & 0 \\ \hline
* & A_\infty - \lambda M_\infty & \\ \hline
* & * & A_\epsilon - M_\epsilon
\end{array}\right], \qquad (3.10)
$$

where

(i) $A_\infty - \lambda M_\infty$ *is a regular pencil containing the infinite elementary divisors of* $A - \lambda M$,

(*ii*) $A_\epsilon - \lambda M_\epsilon$ *and* $A_\eta - \lambda M_\eta$ *are singular pencils containing the Kronecker row and column structures respectively.*

$$A_y - \lambda M_y = \begin{bmatrix} \begin{array}{c|c} 0 & \begin{matrix} 0 & I_{e_k} & & \\ & \ddots & \ddots & \\ & & 0 & I_{e_k} \end{matrix} \\ \hline \begin{matrix} 0 & & \\ I_{e_k} & \ddots & \\ & \ddots & 0 \\ & & I_{e_k} \end{matrix} & 0 \\ \hline \end{array} & & 0 & \\ & \ddots & & \\ 0 & & \begin{array}{|ccc|} \hline 0 & 0 & I_{e_2} \\ 0 & 0 & 0 \\ I_{e_2} & 0 & 0 \\ \hline \end{array} & \\ & & & 0_{e_1} \end{bmatrix}.$$

$$-\lambda \begin{bmatrix} \begin{array}{c|c} 0 & \begin{matrix} I_{e_k} & 0 & & \\ & \ddots & \ddots & \\ & & I_{e_k} & 0 \end{matrix} \\ \hline \begin{matrix} I_{e_k} & & \\ 0 & \ddots & \\ & \ddots & I_{e_k} \\ & & 0 \end{matrix} & 0 \\ \hline \end{array} & & 0 & \\ & \ddots & & \\ 0 & & \begin{array}{|ccc|} \hline 0 & I_{e_2} & 0 \\ I_{e_2} & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array} & \\ & & & 0_{e_1} \end{bmatrix}.$$

FIG. 11.

Proof. Using Lemmas 3.2 and 3.3 and Proposition 3.4, and applying Algorithms 4.1 and 4.5 in [10] to the pencil $A_x - \lambda M_x$, we obtain the desired result. ∎

From above corollary and Lemma 3.1 it clearly follows that

COROLLARY 3.6. *$A - \lambda M$ is strictly equivalent to*

$$\begin{bmatrix} A_f - \lambda M_f & & & \\ * & A_\eta - \lambda M_\eta & & \\ * & * & A_\infty - \lambda M_\infty & \\ * & * & * & A_\epsilon - \lambda M_\epsilon \end{bmatrix}. \tag{3.11}$$

REMARK 3.7

(1) If $A - \lambda M$ is a complex singular pencil, we also have a revision of the step (3.1), replacing the spectral decomposition by a SVD as in step 1′ in Remark 2.5(1), and then we can construct an Algorithm 3.1′ (say) with the above elementary step.

(2) Indeed, when $\det(A - \lambda M) \equiv 0$, the eigenvalue problem $A - \lambda M$ is currently regarded as "ill posed" [9, 10, 12, 15]. Algorithm 3.1, Lemma 3.1, and Corollary 3.5, though show that it can be possible to reduce such pencils to the form (3.11) and thus extract a "finite part" $A_f - \lambda M_f$ with only finite eigenvalues. We do not want to emphasize that this part is well conditioned. On the contrary, almost any perturbation of a square singular pencil will turn this pencil to a regular one. For example, the pencil

$$\begin{bmatrix} 0 & \lambda & 1 \\ \lambda & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

has Kronecker blocks L_1 and L_1^T. A possible perturbation of this pencil is, e.g.

$$\begin{bmatrix} 0 & \lambda & 1 \\ \lambda & -\epsilon_0 & \epsilon_1/2 \\ 1 & \epsilon_1/2 & -\epsilon_3\lambda - \epsilon_2 \end{bmatrix}$$

(with $|\epsilon_i|$ smaller than machine precision). Its determinant is $\epsilon_3\lambda^3 + \epsilon_2\lambda^2 + \epsilon_1\lambda + \epsilon_0$. Up to a scalar factor we can thus construct any polynomial and also any characteristic roots (infinity also, by choosing $\epsilon_3 = 0$). Nevertheless, using the QZ or HR algorithm on a "finite part" $A_f - \lambda M_f$ is more reasonable than using it on the whole pencil $A - \lambda M$, since the QZ or QR algorithm is not able to distinguish "regular" eigenvalues from "fake" ones. Worse, it is possible that none of the ratios corresponds to a "regular" eigenvalue, as shown in the following example: The symmetric pencil

$$A - \lambda M = \left[\begin{array}{c|c|cc} 0 & 0 & \lambda & 1 \\ \hline 0 & \boxed{2-\lambda} & 0 & 0 \\ \hline \lambda & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

has Kronecker blocks L_1 and L_1^T with size one and finite eigenvalue 2. But it has apparent eigenvalues 1/0, 0/0, 0/0, 1/0 detected by the QZ algorithm on the pencil $\Theta(A-\lambda M)$, where $\Theta=[e_n,\dots,e_1]$.

(3) The cost of computations in Algorithm 3.1 is much lower than in Van Dooren's Algorithms 4.1 and 4.5 in [10], and the symmetry of the current matrix is also preserved.

4. CONCLUSION

In this paper we have developed some algorithms for the computation of the Kronecker canonical form of a singular pencil. The method has a lower complexity (i.e. number of multiplications), and so is cheaper than [8, 10], when the size of the singularity is much smaller than the dimension of the pencils. Though some nonunitary transformations are utilized in our algorithms, the norms of these transformations are bounded under a tolerance. A combination of the QZ or HR algorithm and our algorithms can be used for the determination of the Kronecker structure of a symmetric pencil. Since the eigenvalue problem for a singular pencil is an ill-conditioned problem, i.e., any small perturbation may affect the structure of this pencil, so we cannot guarantee that the computed structure indeed corresponds to pencil; but for the regular pencil it is relatively stable [Remark 2.5(2)].

Numerical experience with our methods is still lacking. However, the main aim of this paper is to expose the relations between Kronecker indices and the symmetry of an arbitrary symmetric pencil.

I thank Professor L. Elsner for many helpful discussions and comments.

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Received 28 February 1986; final manuscript accepted 17 June 1987